Exercise 5A

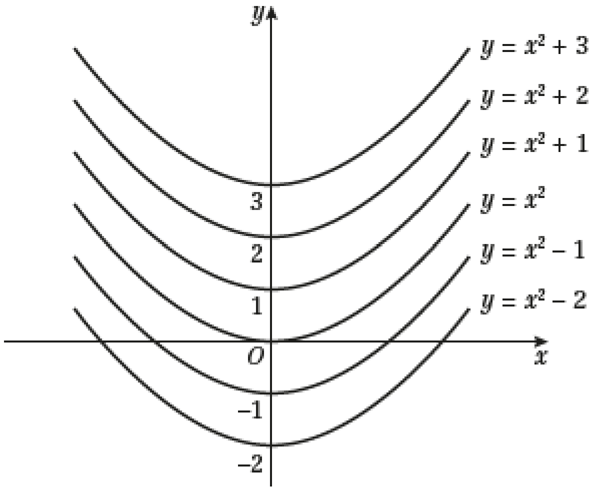
1 Integrating both sides of the equation and including a constant:

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow y = \int 2x \, dx$$

$$\Rightarrow y = x^2 + c \quad \text{where } c \text{ is a constant}$$

The family of solution curves are parabola.
Sketching the solution curves for $c = -2, -1, 0, 1, 2$ and 3 gives:



2 Separating the variables and integrating:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

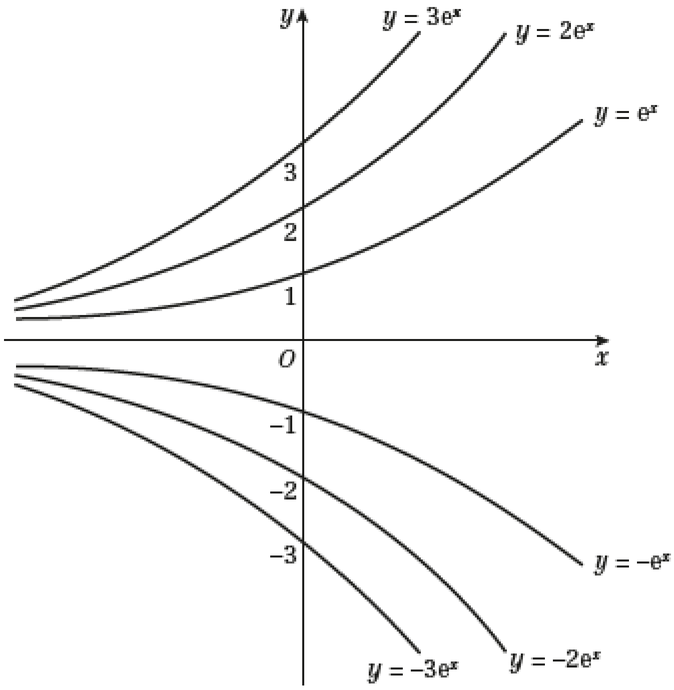
$$\Rightarrow \int \frac{1}{y}\,\mathrm{d}y = \int 1\,\mathrm{d}x$$

$$\Rightarrow \ln y = x + c \quad \text{where } c \text{ is a constant}$$

$$\Rightarrow y = \mathrm{e}^{x+c} = \mathrm{e}^{c} \times \mathrm{e}^{x}$$

$$\Rightarrow y = A\mathrm{e}^{x} \quad \text{where } A \text{ is a constant } (A = \mathrm{e}^{c})$$

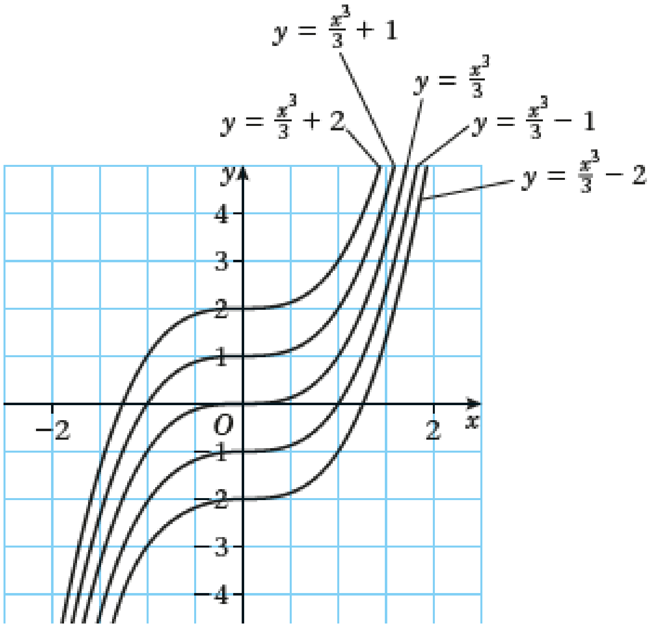
The family of solution curves are exponential curves.
Sketching the solution curves for $A = -3, -2, -1, 1, 2$ and 3 gives:



3 $\frac{dy}{dx} = x^2$

$\int dy = \int x^2 dx$

$y = \frac{1}{3}x^3 + c$



4 $\frac{dy}{dx} = \frac{1}{x}$

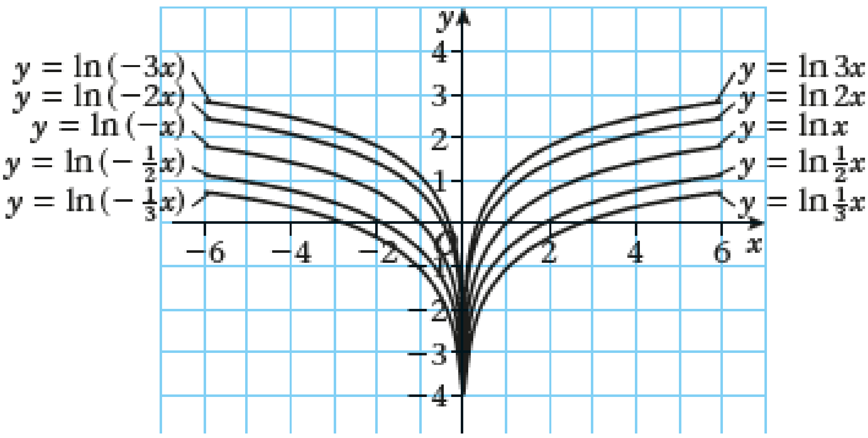
$\int dy = \int \frac{1}{x} dx$

$y = \ln x + c$

Let $c = \ln A$

$y = \ln x + \ln A$

$= \ln Ax$



5 Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\Rightarrow \ln|y| = 2\ln|x| + c$$

Expressing the constant as $\ln|A|$ and simplifying using the laws of logarithms:

$$\ln|y| = 2\ln|x| + \ln|A|$$

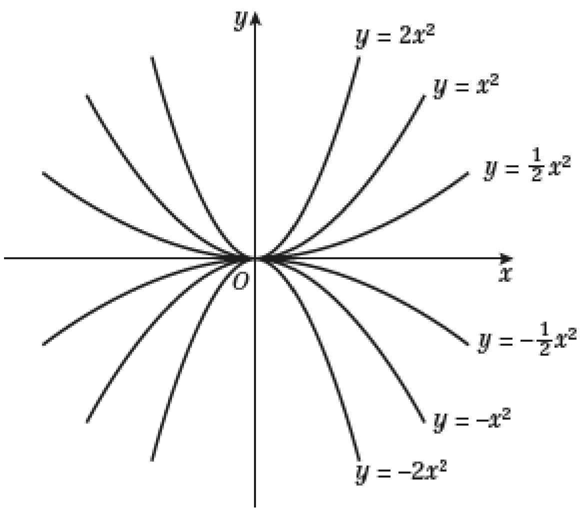
$$\Rightarrow \ln|y| = \ln x^2 + \ln|A| \qquad \text{using the power law}$$

$$\Rightarrow \ln|y| = \ln|A|x^2 \qquad \text{using the multiplication law}$$

$$\Rightarrow y = Ax^2$$

The family of solution curves are parabola.

Sketching the solution curves for $A = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2$ gives:



6 Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{x}{y}$$

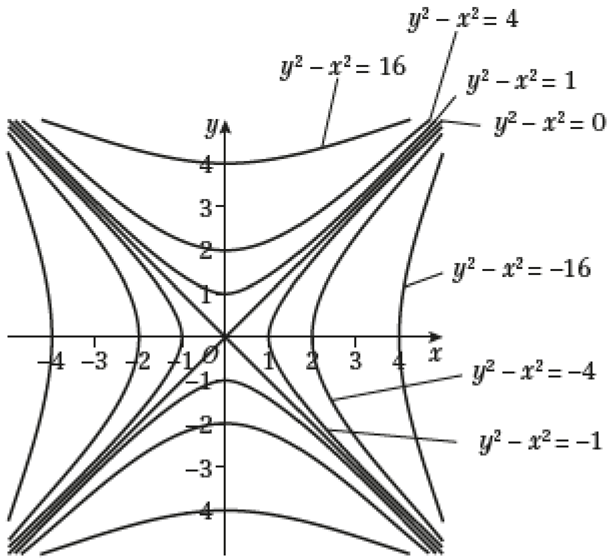
$$\Rightarrow \int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c \quad \text{or} \quad y^2 - x^2 = 2c$$

$y^2 - x^2 = 0 \Rightarrow (y-x)(y+x) = 0$, and the graph of this equation are straight lines $y = x$ and $y = -x$

$y^2 - x^2 = 2c$ for $c \neq 0$ is a hyperbola with asymptotes $y = x$ and $y = -x$

Sketching some of the solution curves gives:



7 $\dfrac{dy}{dx} = e^{y}$

$\int e^{-y}dy = \int dx$

$-e^{-y} = x + c$

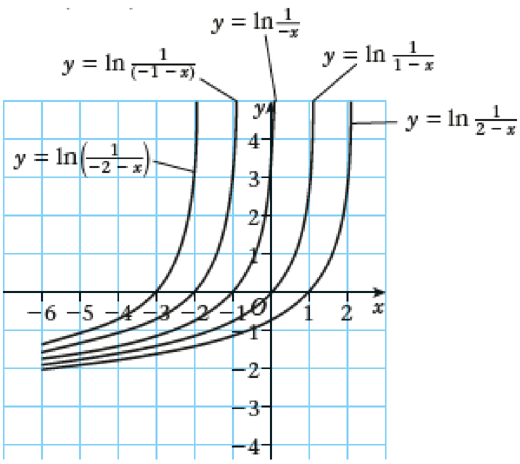
$e^{-y} = -x - c$

$-y = \ln(-x - c)$

$y = -\ln(-x - c)$

$= \ln(-x - c)^{-1}$

$= \ln\dfrac{1}{(-x - c)}$



8 $\dfrac{dy}{dx} = \dfrac{y}{x(x+1)}$

$\int \dfrac{1}{y}\,dy = \int \dfrac{1}{x(x+1)}\,dx$

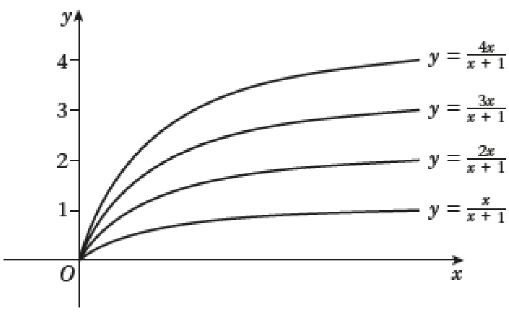
$= \int \dfrac{1}{x}\,dx - \int \dfrac{1}{x+1}\,dx$

$\ln y = \ln x - \ln(x+1) + c$

Let $c = \ln B$

$\ln y = \ln x + \ln(x+1)^{-1} + \ln B$

$= \ln\left(\dfrac{Bx}{x+1}\right)$



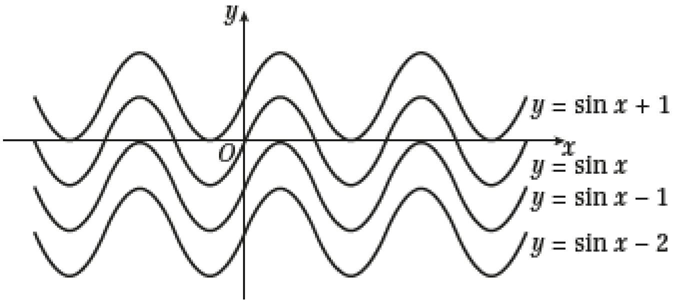
9 $\frac{dy}{dx} = \cos x$

$\Rightarrow y = \sin x + c$

The family of solution curves are sin curves.

The graph of $y = \sin x + c$ is a translation of $y = \sin x$ by the vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$

Sketching some of the solution curves gives:



10 Separating the variables and integrating:

$\frac{dy}{dx} = y \cot x \quad 0 < x < \pi$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{\cos x}{\sin x} dx$

$\Rightarrow \ln|y| = \ln|\sin x| + c$ integrating $\frac{\cos x}{\sin x}$ using the reverse chain rule

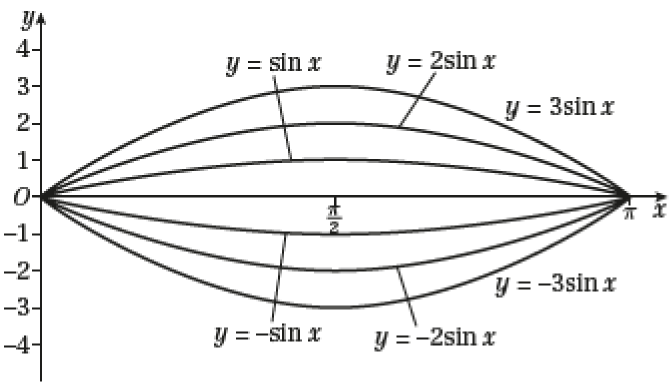
Expressing the constant as $\ln|A|$ and simplifying using the laws of logarithms:

$\ln|y| = \ln|\sin x| + \ln|A|$

$\Rightarrow \ln|y| = \ln|A \sin x|$

$\Rightarrow y = A \sin x$

The family of solution curves are sin curves for $0 < x < \pi$ with varying amplitudes.
Sketching some of the solution curves gives:

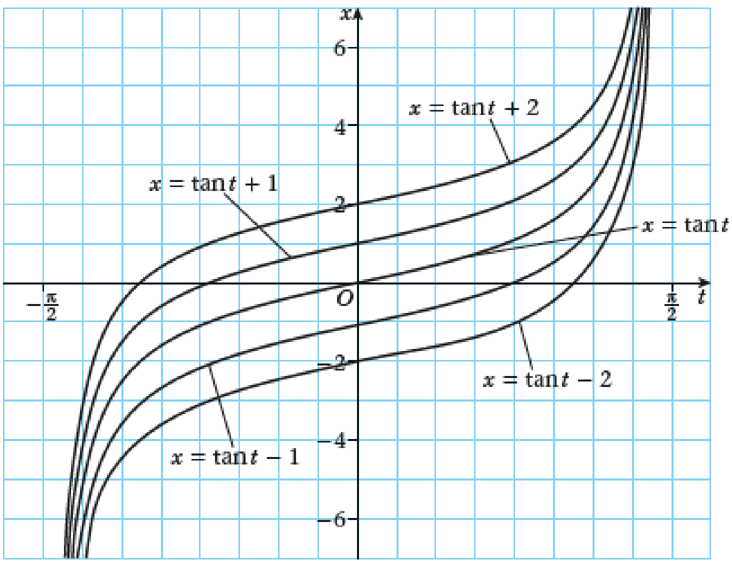


11 $\frac{dy}{dt} = \sec^2 t$ where $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\frac{d}{dt}(\tan t) = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right)$$
$$= \frac{\cos t \cos t + \sin t \sin t}{\cos^2 t}$$
$$= \frac{1}{\cos^2 t}$$
$$= \sec^2 t$$

$$\int dy = \int \sec^2 t \, dt$$

$$y = \tan t + c$$



12 $\frac{dx}{dt} = x(1-x)$ where $0 < x < 1$

$$\int \frac{1}{x(1-x)}\,dx = \int dt$$

$$\int \frac{1}{x}\,dx + \int \frac{1}{1-x}\,dx = \int dt$$

$$\int \frac{1}{x}\,dx - \int \frac{-1}{1-x}\,dx = \int dt$$

$$\ln x - \ln(1-x) = t + B$$

$$\ln\left(\frac{x}{1-x}\right) = t + B$$

$$\frac{x}{1-x} = e^{t+B}$$

$$= e^t e^B$$

$$= Ae^t \quad \text{where } A = e^B$$

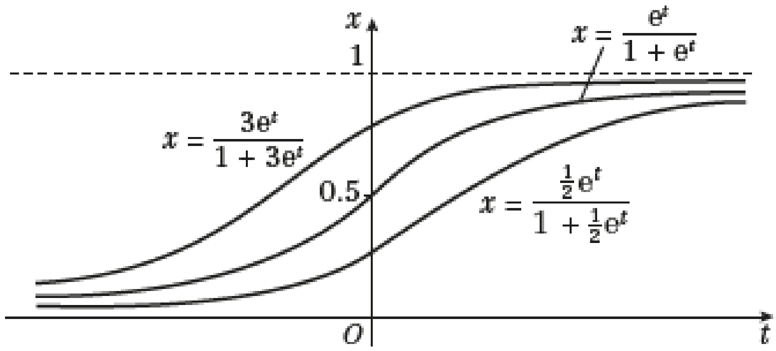
$$x = (1-x)Ae^t$$

$$= Ae^t - Axe^t$$

$$x + Axe^t = Ae^t$$

$$x(1 + Ae^t) = Ae^t$$

$$x = \frac{Ae^t}{1 + Ae^t}$$



13 $\frac{dy}{dx} = \frac{y}{2x}$

$\int \frac{1}{y} \, dy = \frac{1}{2} \int \frac{1}{x} \, dx$

$\ln y = \frac{1}{2} \ln x + c$

$= \ln \sqrt{x} + \ln B \quad \text{where } c = \ln B$

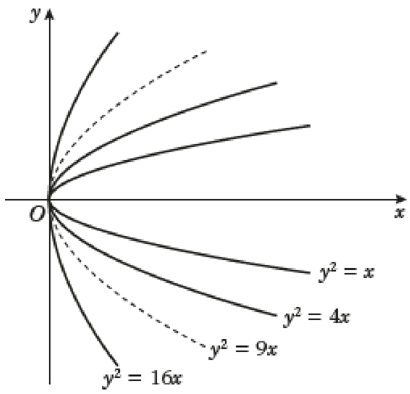
$= \ln B\sqrt{x}$

$y = B\sqrt{x}$

$y^2 = B^2 x$

$= 4ax \quad \text{where } B^2 = 4a$

a



b For (1, 3)

$3^2 = 4a(1)$

$a = \frac{9}{4}$

$y^2 = 9x$

14 $\frac{dy}{dx} = -\frac{xy}{9-x^2}$

$\int \frac{1}{y}\,dy = \int \frac{-x}{9-x^2}\,dx$

$= \frac{1}{2}\int \frac{-2x}{9-x^2}\,dx$

$\ln y = \frac{1}{2}\ln(9-x^2) + c$

$= \ln(9-x^2)^{\frac{1}{2}} + \ln B$

$= \ln B(9-x^2)^{\frac{1}{2}}$

$y = B(9-x^2)^{\frac{1}{2}}$

$y^2 = B^2(9-x^2)$

$= 9B^2 - B^2x^2$

$y^2 + B^2x^2 = 9B^2$

Let $k = B^2$

$y^2 + kx^2 = 9k$

a For the point (2, 5)

$5^2 + k(2)^2 = 9k$

$k = 5$

$y^2 + 5x^2 = 45$

b